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The Stokes-flow drag on prolate and oblate spheroids during axial translatory accelerations

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Stokes's linearized equations of motion are used to calculate the flow field generated by a spheroid executing axial translatory oscillations in an infinite, otherwise still, incompressible, viscous fluid. The flow field, expressed in terms of spheroidal wave functions of order one, is used to develop general expressions for the drag on oscillating prolate and oblate spheroids. Formulae for the approximate drag, useful in making calculations, are obtained for small values of an oscillation parameter. These formulae reduce to the Stokes result in the limit when the spheroid becomes a sphere and the steady-state drag for a spheroid as the frequency of oscillation becomes zero. The fluid forces on spheroids of various shapes are compared graphically. The approximate formulae for the drag are integrated over all frequencies to obtain formulae for the drag on spheroids executing general axial translatory accelerations. The fluid resistance on the spheroid is expressed as the sum of an added mass effect, a steady-state drag and an effect due to the history of the motion. A table of added mass, viscous and history coefficients is given.

1. Introduction

Stokes-flow theory has been used to estimate the drag on spheres (Stokes 1851), ellipsoids (Oberbeck 1876) and axially symmetric bodies (Payne & Pell 1960) moving with constant velocity through an otherwise still, viscous, incompressible fluid of infinite extent. Such an estimate is found to be valid provided the Reynolds number is less than *c*. 0·1. For unsteady motion Stokes-flow theory was used by Stokes (1851) in his classic study of a sphere executing translatory oscillations and by Basset (1888), who calculated the drag on a sphere moving with an arbitrary acceleration along a rectilinear path. Although they are somewhat scant, the experimental reports by Carstens (1952), Odar (1962) and Mockros & Lai (1969) indicate a much larger range of applicability, as expected, for theoretical Stokes-flow drag in the case of accelerating bodies than in the case of steady motions.

This paper presents the Stokes-flow theory for the flow field and drag for a spheroid executing axial translatory oscillations and the drag on a spheroid executing arbitrary axial translatory accelerations. Kanwal (1955) analysed the

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Stokes flow generated by an oscillating spheroid and obtained the general solution for the Stokes stream function in terms of a series of spheroidal wave functions of the first order. He was, however, unable to determine the constants of integration in his solution numerically because of a lack of tables for spheroidal wave functions at that time. The present paper proceeds further and, by investigating the rather complicated spheroidal wave functions, derives formulae for the drag on accelerating spheroids.

Since the analysis for the oblate spheroid motion is parallel to that of the prolate spheroid case detailed calculations are presented only for the latter geometric shape.

2. The flow field generated by an oscillating prolate spheroid

This section is concerned with the formulation of the mathematical representation for the fluid motion induced by a prolate spheroid executing translatory oscillations in an unlimited, otherwise still, incompressible, viscous fluid. The prolate spheroid is defined by $(x/b)^2+(y/b)^2+(z/a)^2 = 1$, the oscillation is assumed parallel to the z direction, and the origin of the co-ordinate system is the instantaneous position of the centre of the prolate spheroid. Prolate spheroidal co-ordinates (ξ, η, ϕ) are natural and the fluid motion is the same in every meridian plane $\phi =$ constant. If $\varpi^2 = x^2+y^2$ and $c^2 = a^2-b^2$, the prolate spheroidal co-ordinates ξ and η are defined by

$$z+i\varpi = c\cosh(\xi+i\eta). \tag{2.1}$$

If, for brevity, $\lambda \equiv \cosh\xi$ and $\zeta \equiv \cos\eta$, the transformation (2.1) leads to the relations

$$z = c\lambda\zeta, \quad \varpi = c[(\lambda^2-1)(1-\zeta^2)]^{\frac{1}{2}}. \tag{2.2}$$

The prolate spheroid is thus specified as $\lambda = \lambda_0 = a/c$.

In a prolate spheroidal co-ordinate system the Stokes-flow equation of motion, in terms of the Stokes stream function Ψ, is (Happel & Brenner 1965, p. 104; Kanwal 1955)

$$E^2\left(E^2-\frac{1}{\nu}\frac{\partial}{\partial t}\right)\Psi = 0, \tag{2.3}$$

in which ν is the kinematic viscosity of the fluid and

$$E^2 = \frac{1}{c^2(\lambda^2-\zeta^2)}\left[(\lambda^2-1)\frac{\partial^2}{\partial\lambda^2}+(1-\zeta^2)\frac{\partial^2}{\partial\zeta^2}\right]. \tag{2.4}$$

The velocity components are related to the Stokes stream function by

$$\left.\begin{aligned} v_\lambda &= \frac{1}{c^2(\lambda^2-1)^{\frac{1}{2}}(\lambda^2-\zeta^2)^{\frac{1}{2}}}\frac{\partial\Psi}{\partial\zeta},\\ v_\zeta &= \frac{1}{c^2(1-\zeta^2)^{\frac{1}{2}}(\lambda^2-\zeta^2)^{\frac{1}{2}}}\frac{\partial\Psi}{\partial\lambda}.\end{aligned}\right\} \tag{2.5}$$

The boundary conditions at the surface of the prolate spheroid are

$$\left.\begin{aligned} \partial\Psi/\partial\zeta &= c^2(\lambda^2-1)\zeta u(t)\\ \partial\Psi/\partial\lambda &= -c^2\lambda(1-\zeta^2)u(t)\end{aligned}\right\} \quad \text{at} \quad \lambda = \lambda_0, \tag{2.6}$$

where $u(t)$ is the instantaneous velocity of the oscillating prolate spheroid. At infinity the fluid is assumed still and

$$\left.\begin{aligned} v_\lambda &= \frac{1}{c^2(\lambda^2-1)^{\frac{1}{2}}(\lambda^2-\zeta^2)^{\frac{1}{2}}}\frac{\partial\Psi}{\partial\zeta} = 0 \\ v_\zeta &= \frac{1}{c^2(1-\zeta^2)^{\frac{1}{2}}(\lambda^2-\zeta^2)^{\frac{1}{2}}}\frac{\partial\Psi}{\partial\lambda} = 0 \end{aligned}\right\} \quad \text{as} \quad \lambda\to\infty. \tag{2.7}$$

The boundary-value problem (2.3), (2.6) and (2.7) is solved in the domain $-1 \leqslant \zeta \leqslant 1$, $\lambda_0 \leqslant \lambda$ by introducing

$$u(t) = u_0\, e^{-i\omega t} \tag{2.8}$$

for the spheroid motion and

$$\Psi(\lambda, \zeta, t) = \psi(\lambda, \zeta)\, e^{-i\omega t} \tag{2.9}$$

for the stream function. The physically meaningful quantities are, of course, only the real parts. Substitution of (2.8) and (2.9) into (2.3) and (2.6) yields

$$E^2(E^2 + h^2/c^2)\,\psi = 0 \tag{2.10}$$

and

$$\left.\begin{aligned} \partial\psi/\partial\lambda &= -c^2(1-\zeta^2)\,\lambda u_0 \\ \partial\psi/\partial\zeta &= c^2\zeta(\lambda^2-1)\,u_0 \end{aligned}\right\} \quad \text{at} \quad \lambda = \lambda_0, \tag{2.11}$$

where $h^2 = ic^2\omega/\nu$.

The general solution of (2.10) can be written as

$$\psi(\lambda, \zeta) = \psi_1(\lambda, \zeta) + \psi_2(\lambda, \zeta), \tag{2.12}$$

provided that

$$E^2\psi_1 = 0 \tag{2.13}$$

and

$$(E^2 + h^2/c^2)\,\psi_2 = 0. \tag{2.14}$$

The appropriate solution of (2.13) is

$$\psi_1(\lambda, \zeta) = \tfrac{1}{2}c^2 u_0 \sum_{r=0}^{\infty} \alpha_r(\lambda^2-1)\,Q'_{r+1}(\lambda)\,(1-\zeta^2)\,P'_{r+1}(\zeta), \tag{2.15}$$

in which the α_r's are constants of integration, $P_{r+1}(\zeta)$ and $Q_{r+1}(\lambda)$ represent Legendre functions of the first and second kind, respectively, and the primes denote ordinary differentiation of the functions. The appropriate solution of (2.14) is

$$\psi_2(\lambda, \zeta) = \tfrac{1}{2}c^2 u_0 \sum_{n=1}^{\infty} \beta_n(\lambda^2-1)^{\frac{1}{2}}\,R^{(3)}_{1n}(h, \lambda)\,(1-\zeta^2)^{\frac{1}{2}}\,S_{1n}(h, \zeta), \tag{2.16}$$

in which the β_n's are constants of integration, the $R^{(3)}_{1n}(h, \lambda)$ are prolate spheroidal radial functions of the third kind of order one and the $S_{1n}(h, \zeta)$ are prolate spheroidal angle functions of the first kind of order one. The angle functions $S_{1n}(h, \zeta)$, which are regular at $\zeta = \pm 1$, can be expressed as

$$S_{1n}(h, \zeta) = \sum_{r=0,1}^{\infty}{}' \, d^{1n}_r(h)\,P^1_{r+1}(\zeta), \tag{2.17}$$

where P^1_{r+1} is an associated Legendre function of the first kind of order one and the prime over the summation sign indicates that the summation is over only even values of r when n is odd and over only odd values of r when n is even. The expansion coefficients d_r^{1n} depend on h only and are expanded in powers of h when h is small. The series in (2.17) converges absolutely in the domain $-1 \leqslant \zeta \leqslant 1$. The radial functions $R_{1n}^{(3)}(h, \lambda)$ represent spherically converging waves at large distances since the parameter h is complex with a positive imaginary part. The converging functions $R_{1n}^{(3)}$ are chosen for the solution so as to satisfy the boundary condition of (2.7). The asymptotic behaviour of these functions is obtained from the asymptotic expression for the spherical Hankel functions of the second kind. Thus, as $\lambda \rightarrow \infty$,

$$R_{1n}^{(3)}(h, \lambda) \rightarrow (1/h\lambda) \exp\{+i[h\lambda - \tfrac{1}{2}(n+1)\pi]\}. \tag{2.18}$$

The functions $R_{1n}^{(3)}(h, \lambda)$ are related to the prolate spheroidal radial functions of the first kind, $R_{1n}^{(1)}(h, \lambda)$, and the second kind, $R_{1n}^{(2)}(h, \lambda)$, of order one by

$$R_{1n}^{(3)}(h, \lambda) = R_{1n}^{(1)}(h, \lambda) + iR_{1n}^{(2)}(h, \lambda). \tag{2.19}$$

When the parameter h is not too large the functions $R_{1n}^{(1)}(h, \lambda)$ and $R_{1n}^{(2)}(h, \lambda)$ can be expressed in terms of a series of associated Legendre functions. (Some mathematical properties of spheroidal wave functions are given by Lai (1969).)

Thus, the appropriate general solution for the Stokes stream function is

$$\psi(\lambda, \zeta) = \tfrac{1}{2}c^2 u_0 \left[\sum_{r=0}^{\infty} \alpha_r (\lambda^2 - 1)\, Q'_{r+1}(\lambda)\, (1-\zeta^2)\, P'_{r+1}(\zeta) + \sum_{n=1}^{\infty} \beta_n (\lambda^2-1)^{\frac{1}{2}} R_{1n}^{(3)}(h, \lambda)\, (1-\zeta^2)^{\frac{1}{2}} S_{1n}(h, \zeta)\right]. \tag{2.20}$$

Using the expansion (2.17) for $S_{1n}(h, \zeta)$ and rearranging the second series in (2.20), the solution for the Stokes stream function can be rewritten as

$$\psi(\lambda, \zeta) = \tfrac{1}{2}c^2 u_0 \sum_{r=0}^{\infty} \left\{\left[\alpha_r (\lambda^2 - 1)\, Q'_{r+1}(\lambda) + \sum_{n=1,2}^{\infty}{}' \beta_n (\lambda^2 - 1)^{\frac{1}{2}} R_{1n}^{(3)}(h, \lambda)\, d_r^{1n}(h)\right] (1-\zeta^2)\, P'_{r+1}(\zeta)\right\}, \tag{2.21}$$

in which the prime over the last summation sign indicates that the summation is over only odd values of n when r is even and over only even values of n when r is odd.

The constants of integration α_r and β_n are determined by using the boundary condition given in (2.11). Thus,

$$\left.\begin{aligned} -\lambda_0(1-\zeta^2) &= \tfrac{1}{2} \sum_{r=0}^{\infty} \left\{\left[\alpha_r (r+1)(r+2)\, Q_{r+1}(\lambda_0) + \sum_{n=1,2}^{\infty}{}' \beta_n \overline{R_{1n}^{(3)}(h, \lambda_0)'}\, d_r^{1n}(h)\right] (1-\zeta^2)\, P'_{r+1}(\zeta)\right\}, \\ (\lambda_0^2 - 1)\, \zeta &= \tfrac{1}{2} \sum_{r=0}^{\infty} \left\{\left[\alpha_r (\lambda_0^2 - 1)\, Q'_{r+1}(\lambda_0) + \sum_{n=1,2}^{\infty}{}' \beta_n (\lambda_0^2 - 1)^{\frac{1}{2}} R_{1n}^{(3)}(h, \lambda_0)\, d_r^{1n}(h)\right] [-(r+1)(r+2)\, P_{r+1}(\zeta)]\right\}, \end{aligned}\right\} \tag{2.22}$$

where $$\overline{R_{1n}^{(3)}(h,\lambda_0)'} = \frac{d}{d\lambda}[(\lambda^2-1)^{\frac{1}{2}} R_{1n}^{(3)}(h,\lambda)]_{\lambda_0}.$$

Because of the orthogonality of Legendre functions (2.22) indicates that

$$\tfrac{1}{2}\left[\alpha_r(r+1)(r+2)\,Q_{r+1}(\lambda_0) + \sum_{n=1,2}^{\infty}{}' \beta_n d_r^{1n}\overline{R_{1n}^{(3)}(h,\lambda_0)'}\right] = \begin{cases} -\lambda_0 & (r=0), \quad (2.23a)\\ 0 & (r=1,2,3,\ldots), \quad (2.23b)\end{cases}$$

$$\tfrac{1}{2}(r+1)(r+2)\left[\alpha_r(\lambda_0^2-1)\,Q'_{r+1}(\lambda_0) + \sum_{n=1,2}^{\infty}{}' \beta_n d_r^{1n}(\lambda_0^2-1)^{\frac{1}{2}} R_{1n}^{(3)}(h,\lambda_0)\right] = \begin{cases} -(\lambda_0^2-1) & (r=0), \quad (2.23c)\\ 0 & (r=1,2,3,\ldots). \quad (2.23d)\end{cases}$$

The α_r s are eliminated from (2.23) and after some manipulation we have

$$\sum_{n=1,2}^{\infty}{}' \beta_n d_r^{1n}[Q_{r+1}^{1}(\lambda_0)\,R_{1n}^{(3)\prime}(h,\lambda_0) - Q_{r+1}^{1\prime}(\lambda_0)\,R_{1n}^{(3)}(h,\lambda_0)] = \begin{cases} 2/(\lambda_0^2-1) & (r=0)\\ 0 & (r=1,2,3,\ldots)\end{cases}. \quad (2.24)$$

where Q_{r+1}^{1} are associated Legendre functions of the second kind of order one. The β_n's are determined by solv ing this system of simultaneous linear algebraic equations.

3. The general expression for the drag on an oscillating prolate spheroid

The force exerted on an axially symmetric body moving along its axis of revolution is given by (Happel & Brenner 1965, p. 114)

$$F_z = \pi \int \varpi^2 \frac{\partial p}{\partial \eta}\, d\eta - 2\pi\mu \int \frac{\partial \varpi}{\partial \xi} E^2\Psi\, d\eta, \quad (3.1)$$

where p and μ denote the pressure and absolute viscosity of the fluid, respectively, and the integral is evaluated over the surface of the body. The pressure is determined from the η component of the Stokes-flow form of the Navier–Stokes equations, i.e.

$$\frac{\partial p}{\partial \eta} = \frac{\mu}{\varpi}\frac{\partial}{\partial \xi}\left[\left(E^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\Psi\right]. \quad (3.2)$$

For the prolate spheroid (3.1) is

$$F_z = c\mu\pi \int_{-1}^{1} \left[(\lambda^2-1)\frac{\partial}{\partial \lambda}\left(E^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\Psi - 2\lambda E^2\Psi\right]_{\lambda_0} d\zeta \quad (3.3)$$

and from (2.9)–(2.14)

$$\left(E^2 - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\Psi = \frac{h^2}{c^2}\psi_1 e^{-i\omega t}, \quad E^2\Psi = -\frac{h^2}{c^2}\psi_2 e^{-i\omega t}. \quad (3.4)$$

Thus $$F_z = \frac{\mu\pi h^2 e^{-i\omega t}}{c}\int_{-1}^{1}\left[(\lambda^2-1)\frac{\partial \psi_1}{\partial \lambda} + 2\lambda\psi_2\right]_{\lambda_0} d\zeta, \quad (3.5)$$

or $$F_z = \tfrac{4}{3}\pi\mu c h^2 u_0 e^{-i\omega t}\left[(\lambda_0^2-1)\,Q_1(\lambda_0)\,\alpha_0 + \sum_{n=1}^{\infty}{}' \beta_n d_0^{1n}\lambda_0(\lambda_0^2-1)^{\frac{1}{2}} R_{1n}^{(3)}(h,\lambda_0)\right], \quad (3.6)$$

in which the summation is over only odd values of n, since the coefficients d_0^{1n} vanish for even values of n. Finally, if (2.23c) and the relations

$$h^2 u_0 e^{-i\omega t} = \frac{i\omega c^2}{\nu} u(t) = -\frac{1}{\nu}\frac{du}{dt}, \quad \lambda_0 = \frac{a}{c} = \frac{a}{(a^2-b^2)^{\frac{1}{2}}}$$

are used, the drag on an oscillating prolate spheroid can be expressed as

$$F_z = -\tfrac{4}{3}\pi a b^2 \rho \left[\frac{(\lambda_0^2-1)\,Q_1(\lambda_0)}{1-(\lambda_0^2-1)\,Q_1(\lambda_0)}\right]\frac{du}{dt} - \frac{4\pi\mu c h^2 u}{3Q_1^1(\lambda_0)} \sum_{n=1}^{\infty}{}' \beta_n d_0^{1n}(h)\, R_{1n}^{(3)}(h,\lambda_0), \quad (3.7)$$

where ρ is the fluid density.

Equation (3.7) is the general expression for the drag on the oscillating prolate spheroid. The first term on the right-hand side, which is independent of the fluid viscosity, is the added mass effect. The quantity

$$C_A = [(\lambda_0^2-1)\,Q_1(\lambda_0)]/[1-(\lambda_0^2-1)\,Q_1(\lambda_0)]$$

is the added mass coefficient for a prolate spheroid accelerating along its axis of symmetry and agrees with the known result (Lamb 1932, p. 153). To evaluate the drag numerically, the β_n's must be first determined by solving the infinite system of simultaneous equations, (2.24). For purposes of numerical evaluation the drag is estimated by using expansions in powers of h.

4. A formula for the approximate drag on an oscillating prolate spheroid

The system of equations (2.24) can be separated into two sets of simultaneous equations, one for even values of r and one for odd values of r:

$$\sum_{n=1}^{\infty}{}' \beta_n d_r^{1n}[Q_{r+1}^1 R_{1n}^{(3)\prime} - Q_{r+1}^{1\prime} R_{1n}^{(3)}]_{\lambda_0} = \begin{cases} 2/(\lambda_0^2-1) & (r=0), \\ 0 & (r=2,4,6,\ldots), \end{cases} \quad (4.1)$$

$$\sum_{n=2}^{\infty}{}' \beta_n d_r^{1n}[Q_{r+1}^1 R_{1n}^{(3)\prime} - Q_{r+1}^{1\prime} R_{1n}^{(3)}]_{\lambda_0} = 0 \quad (r=1,3,5,\ldots). \quad (4.2)$$

When n is even, the constants β_n, which satisfy the system of homogeneous equations (4.2), are not required in the computation of the drag. In (3.7) the summation is over odd values of n and therefore only the solution of (4.1) is needed.

When h is small ($|h| < 10$, see Flammer 1957, p. 59), the coefficients $d_r^{1n}(h)$ and the radial functions $R_{1n}^{(3)}(h,\lambda)$ can be expanded in power series of h. Using these expansions (developed in Lai 1969), in powers of h, for $d_r^{1n}(h)$ as well as for $R_{1n}^{(3)}(h,\lambda_0)$ the first two equations of the system equation (4.1) are

$$\underset{(0)}{\beta_1 d_0^{11}}[Q_1^1 R_{11}^{(3)\prime} \underset{(0)}{-} Q_1^{1\prime} R_{11}^{(3)}]_{\lambda_0} + \underset{(2)}{\beta_3 d_0^{13}}[Q_1^1 R_{13}^{(3)\prime} \underset{(-4)}{-} Q_1^{1\prime} R_{13}^{(3)}]_{\lambda_0}$$

$$+\underset{(4)}{\beta_5 d_0^{15}}[Q_1^1 R_{15}^{(3)\prime} \underset{(-6)}{-} Q_1^{1\prime} R_{15}^{(3)}]_{\lambda_0} + \ldots = 2/(\lambda_0^2 \underset{(0)}{-} 1), \quad (4.3)$$

$$\underset{(2)}{\beta_1 d_2^{11}}[Q_3^1 R_{11}^{(3)\prime} \underset{(-2)}{-} Q_3^{1\prime} R_{11}^{(3)}]_{\lambda_0} + \underset{(0)}{\beta_3 d_2^{13}}[Q_3^1 R_{13}^{(3)\prime} \underset{(-2)}{-} Q_3^{1\prime} R_{13}^{(3)}]_{\lambda_0}$$

$$+\underset{(2)}{\beta_5 d_2^{15}}[Q_3^1 R_{15}^{(3)\prime} \underset{(-6)}{-} Q_3^{1\prime} R_{15}^{(3)}]_{\lambda_0} + \ldots = \underset{(0)}{0}, \quad (4.4)$$

where (r) indicates that the term above is of order h^r. In (4.3), the right-hand side is of order h^0; hence, the lowest orders in h for β_1 and β_3 are h^0 and h^2, respectively. The order of β_5, estimated from (4.4), is h^4. Thus, $\beta_1, \beta_2, \beta_5, \ldots$ are of order $h^0, h^2, h^4, \ldots$, respectively.

Based on these estimated orders, the constants β_n are expanded in power series of h, i.e.

$$\left.\begin{aligned} \beta_1 &= a_{10} + a_{11}h + a_{12}h^2 + \ldots, \\ \beta_3 &= \qquad\qquad a_{32}h^2 + a_{33}h^3 + \ldots, \\ \beta_5 &= \qquad\qquad\qquad\qquad a_{54}h^4 + a_{55}h^5 + \ldots. \end{aligned}\right\} \tag{4.5}$$

The coefficients a_{nr} in (4.5) are determined as follows. The substitution of (4.5) into the system given by (4.1) makes each equation of system (4.1) a power series in h. The simultaneous consideration of the zeroth-order terms from each equation of the power-series version of (4.1) permits the calculation of the lowest order coefficient in each β_n expansion, i.e. $a_{10}, a_{32}, a_{54}, \ldots$. The next lowest order coefficients in the β_n expansions, i.e. $a_{11}, a_{33}, a_{55}, \ldots$, are determined from the simultaneous solution of the terms in the power-series version of (4.1) that are of order h.

The power-series versions of the first two equations of (4.1) are

$$\begin{aligned} &(a_{10}+a_{11}h+a_{12}h^2+\ldots)\left(1-\tfrac{1}{50}h^2+\ldots\right)\left(\frac{+3i}{2h^2}\right)\{\tfrac{1}{3}[Q_1^1(\lambda Q_0^{1\prime}+Q_0^1)-Q_1^{1\prime}\lambda Q_0^1]h^2 \\ &\quad -\tfrac{1}{75}[Q_1^1Q_3^{1\prime}-Q_1^{1\prime}Q_3^1]h^2-\tfrac{2}{9}i[Q_1^1P_1^{1\prime}-Q_1^{1\prime}P_1^1]h^3+\ldots\}_{\lambda_0} \\ &\quad +(a_{32}h^2+a_{33}h^3+\ldots)\left(\tfrac{6}{175}h^2+\ldots\right)\left(\frac{+525i}{8h^4}\right) \\ &\qquad \times\{[Q_1^1Q_3^{1\prime}-Q_1^{1\prime}Q_3^1]+\tfrac{79}{1350}[Q_1^1Q_3^{1\prime}-Q_1^{1\prime}Q_3^1]h^2-\tfrac{2}{189}[Q_1^1Q_5^{1\prime}-Q_1^{1\prime}Q_5^1]h^2+\ldots\}_{\lambda_0} \\ &\quad +(a_{54}h^4+a_{55}h^5+\ldots)\left(\frac{h^4}{4851}+\ldots\right)\left(\frac{+218295i}{16h^6}\right) \\ &\qquad \times\{[Q_1^1Q_5^{1\prime}-Q_1^{1\prime}Q_5^1]+\ldots\}_{\lambda_0}+\ldots=\frac{2}{\lambda_0^2-1}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &(a_{10}+a_{11}h+\ldots)\left(\frac{-h^2}{75}+\ldots\right)\left(\frac{+3i}{2h^2}\right)\{[Q_3^1Q_1^{1\prime}-Q_3^{1\prime}Q_1^1]+\tfrac{1}{3}[Q_3^1(\lambda Q_0^{1\prime}+Q_0^1)-Q_3^{1\prime}\lambda Q_0^1]h^2 \\ &\quad -\tfrac{3}{10}[Q_3^1Q_1^{1\prime}-Q_3^{1\prime}Q_1^1]h^2-\tfrac{2}{9}i[Q_3^1P_1^{1\prime}-Q_3^{1\prime}P_1^1]h^3+\ldots\}_{\lambda_0} \\ &\quad +(a_{32}h^2+a_{33}h^3+\ldots)\left(1+\frac{13h^2}{1350}+\ldots\right)\left(\frac{+525i}{8h^4}\right)\{\tfrac{6}{175}[Q_3^1Q_1^{1\prime}-Q_3^{1\prime}Q_1^1]h^2 \\ &\quad -\tfrac{2}{189}[Q_3^1Q_5^{1\prime}-Q_3^{1\prime}Q_5^1]h^2+\ldots\}_{\lambda_0}+(a_{54}h^4+a_{55}h^5+\ldots)\left(\frac{5h^2}{297}+\ldots\right) \\ &\qquad \times\left(\frac{+218295i}{16h^6}\right)\{[Q_3^1Q_5^{1\prime}-Q_3^{1\prime}Q_5^1]+\tfrac{38}{4563}[Q_3^1Q_5^{1\prime}-Q_3^{1\prime}Q_5^1]h^2 \\ &\quad -\tfrac{15}{1859}[Q_3^1Q_7^{1\prime}-Q_3^{1\prime}Q_7^1]h^2+\ldots\}_{\lambda_0}+\ldots=0. \end{aligned} \tag{4.7}$$

Comparing the coefficients of zeroth power in h in (4.6) and (4.7) yields

$$+\tfrac{1}{2}ia_{10}[Q_1^1(\lambda Q_0^{1\prime}+Q_0^1)-Q_1^{1\prime}\lambda Q_0^1]_{\lambda_0}-\left(\tfrac{1}{50}ia_{10}-\tfrac{9}{4}ia_{32}\right)[Q_1^1Q_3^{1\prime}-Q_1^{1\prime}Q_3^1]_{\lambda_0}=\frac{2}{\lambda_0^2-1} \tag{4.8}$$

and

$$\left(\frac{-i}{50}a_{10}+\frac{9i}{4}a_{32}\right)[Q_1^1Q_3^{1\prime}-Q_1^{1\prime}Q_3^1]_{\lambda_0}-\left(\frac{25i}{36}a_{32}-\frac{3675i}{16}a_{54}\right)[Q_3^1Q_5^{1\prime}-Q_3^{1\prime}Q_5^1]_{\lambda_0}=0. \tag{4.9}$$

Equations (4.8) and (4.9) are the first two equations of the system of infinite simultaneous equations obtained by equating the zeroth power of h in each equation of the power-series version of (4.1). The solution of these equations for the first few coefficients is

$$\left.\begin{aligned}a_{10}&=\frac{-4i}{(\lambda_0^2-1)[Q_1^1(\lambda Q_0^{1\prime}+Q_0^1)-Q_1^{1\prime}\lambda Q_0^1]_{\lambda_0}}=\frac{-4i}{[\frac{1}{2}(\lambda_0^2+1)\log[(\lambda_0+1)/(\lambda_0-1)]-\lambda_0]},\\ a_{32}&=\tfrac{2}{225}a_{10},\quad a_{54}=\tfrac{8}{297675}a_{10},\ \dots\end{aligned}\right\} \tag{4.10}$$

With the aid of the expressions in (4.10), the next lowest order coefficients in the β_n expansions, i.e. $a_{11}, a_{33}, a_{55}, \dots$, can be found by simultaneously considering the first power of h in each equation of the power-series version of (4.1). The results are

$$\left.\begin{aligned}a_{11}&=\frac{8(\lambda_0^2-1)[Q_1^1P_1^{1\prime}-Q_1^{1\prime}P_1^1]_{\lambda_0}}{3[\frac{1}{2}(\lambda_0^2+1)\log((\lambda_0+1)/(\lambda_0-1))-\lambda_0]^2}=\frac{-16}{3[\frac{1}{2}(\lambda_0^2+1)\log[(\lambda_0+1)/(\lambda_0-1)]-\lambda_0]^2},\\ a_{33}&=\tfrac{2}{225}a_{11},\quad a_{55}=\tfrac{8}{297675}a_{11},\ \dots\end{aligned}\right\} \tag{4.11}$$

Defining
$$\kappa=\tfrac{1}{2}(\lambda_0^2+1)\log[(\lambda_0+1)/(\lambda_0-1)]-\lambda_0, \tag{4.12}$$
the constants of integration β_n are thus found in terms of powers of h to be

$$\left.\begin{aligned}\beta_1&=\frac{-4i}{\kappa}-\frac{16}{3\kappa^2}h+O(h^2),\\ \beta_3&=\frac{2}{225}\left[\frac{-4i}{\kappa}h^2-\frac{16}{3\kappa^2}h^3+O(h^4)\right],\\ \beta_5&=\frac{8}{297675}\left[\frac{-4i}{\kappa}h^4-\frac{16}{3\kappa^2}h^5+O(h^6)\right].\end{aligned}\right\} \tag{4.13}$$

By substituting the expressions for the β_n's equation (4.13), and the series expansions for the d_r^{1n}'s and $R_{1n}^{(3)}$'s into (3.7), an expansion in powers of h for the approximate drag experienced by an oscillating prolate spheroid is found to be

$$F_z=-\tfrac{4}{3}\pi ab^2\rho\left[\frac{(\lambda_0^2-1)Q_1(\lambda_0)}{1-(\lambda_0^2-1)Q_1(\lambda_0)}\right]\frac{du}{dt}-\frac{4\pi\mu ch^2u}{3Q_1^1(\lambda_0)}\times\left\{\frac{1}{h^2}\left[\frac{6}{\kappa}Q_1^1(\lambda_0)-\frac{8i}{\kappa^2}Q_1^1(\lambda_0)h+O(h^2)\right]\right\}. \tag{4.14}$$

Equation (4.14) can also be written as

$$F_z=-\left\{\tfrac{4}{3}\pi ab^2\rho\left[\frac{(\lambda_0^2-1)Q_1(\lambda_0)}{1-(\lambda_0^2-1)Q_1(\lambda_0)}\right]+\frac{32\pi\mu a^2}{3\lambda_0^2\kappa^2}\left(\frac{1}{2\nu\omega}\right)^{\frac{1}{2}}\right\}\frac{du}{dt}-\left[\frac{8\pi\mu a}{\lambda_0\kappa}+\frac{32\pi\mu a^2}{3\lambda_0^2\kappa^2}\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\right]u(t), \tag{4.15}$$

which is correct to order h.

When the frequency $\omega = 0$, that is, for steady motion, (4.15) becomes

$$F_z = -\frac{8\pi\mu a u}{\lambda_0[\frac{1}{2}(\lambda_0^2+1)\log[(\lambda_0+1)/(\lambda_0-1)]-\lambda_0]}, \quad (4.16)$$

the result obtained by Payne & Pell (1960) for the drag on a prolate spheroid moving at constant velocity. When $\lambda_0 \to \infty$, i.e. $a = b$, the prolate spheroid becomes a sphere. By using the asymptotic expansion

$$\tfrac{1}{2}\log\frac{\lambda_0+1}{\lambda_0-1} = \frac{1}{\lambda_0}+\frac{1}{3\lambda_0^3}+\frac{1}{5\lambda_0^5}+\ldots,$$

(4.15) is seen to approach

$$F_z = -\left[\tfrac{2}{3}\pi a^3\rho + 6\pi\mu a^2\left(\frac{1}{2\nu\omega}\right)^{\frac{1}{2}}\right]\frac{du}{dt} - \left[6\pi\mu a + 6\pi\mu a^2\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\right]u, \quad (4.17)$$

which is the classical Stokes solution for an oscillating sphere. When $\lambda_0 = 1$, i.e. $b = 0$, the prolate spheroid reduces to a needle-like body, and the drag, as expected, is zero. This latter result has also been obtained by Aoi (1955) and Breach (1961) in their studies of the steady motion of a prolate spheroid in a viscous fluid.

5. The drag on an oscillating oblate spheroid

For an oblate spheroid

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2} = 1$$

oscillating along the z axis in an unlimited, otherwise still, incompressible, viscous fluid, the analysis is similar to the case of a prolate spheroid. All the equations and solutions for the oblate spheroid can be obtained from those for the prolate spheroid by a simple transformation to imaginary values of parameters and co-ordinates. In this paper only the drag is presented.

The formula for the approximate drag on an oblate spheroid oscillating along its axis of symmetry, i.e. broadside-on, in a viscous fluid is found from (4.15) by substituting $i\lambda$ for λ. The resulting drag is

$$F_z = -\left\{\tfrac{4}{3}\pi a^2 b\rho\left[\frac{(\lambda_0^{*2}+1)q_1(\lambda_0^*)}{1-(\lambda_0^{*2}+1)q_1(\lambda_0^*)}\right]+\frac{32\pi\mu b^2}{3\lambda_0^{*2}\kappa^{*2}}\left(\frac{1}{2\nu\omega}\right)^{\frac{1}{2}}\right\}\frac{du}{dt} - \left\{\frac{8\pi\mu b}{\lambda_0^*\kappa^*}+\frac{32\pi\mu b^2}{3\lambda_0^{*2}\kappa^{*2}}\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\right\}u, \quad (5.1)$$

where $\lambda_0^* = b/c$, $q_1(\lambda_0^*) = 1-\lambda_0^*\cot^{-1}\lambda_0^*$, $\kappa^* = \lambda_0^*-(\lambda_0^{*2}-1)\cot^{-1}\lambda_0^*$, and the rest of the notation is the same as for the prolate spheroid. The functions $q_r(\lambda)$ are called ellipsoidal harmonics and are related to Legendre functions by

$$q_r(\lambda) = i^{r+1}Q_r(i\lambda).$$

The quantity

$$C_A^* = [(\lambda_0^{*2}+1)q_1(\lambda_0^*)]/[1-(\lambda_0^{*2}+1)q_1(\lambda_0^*)]$$

is the added mass coefficient for an oblate spheroid accelerating along its axis of symmetry and agrees with the known result derived from potential-flow theory (Lamb 1932, p. 700).

When the frequency ω is zero, (5·1) becomes the result obtained by Payne & Pell (1960) for steady motion. Using an expansion for $\cot^{-1}\lambda_0^*$ and letting $\lambda_0^* \to \infty$, (5.1) approaches the Stokes result, equation (4.17), as the oblate spheroid approaches a sphere.

If $\lambda_0^* = 0$, i.e. $b = 0$, the oblate spheroid degenerates to a circular disk, and the expression for the drag experienced by a circular disk oscillating along the direction normal to its surface is obtained by letting $\lambda_0^* \to 0$ in (5.1). The result is

$$F_z = -\left[\tfrac{8}{3}a^3\rho + \frac{128\mu a^2}{3\pi}\left(\frac{1}{2\nu\omega}\right)^{\frac{1}{2}}\right]\frac{du}{dt} - \left[16\mu a + \frac{128\mu a^2}{3\pi}\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}\right]u. \tag{5.2}$$

In this expression, the term $\frac{8}{3}a^3\rho(du/dt)$ is the potential-flow solution drag and the term $16\mu au$ is the result obtained by Oberbeck (1876) for the drag on a circular disk moving at constant velocity.

6. The effect of shape on the drag of oscillating spheroids

The formula (4.15) for the drag on a prolate spheroid executing translatory oscillations along its axis of symmetry in an incompressible viscous fluid is derived from an expansion of spheroidal wave functions in a power series of h and is correct to the first order in h. The drag on the spheroid may be compared with that on a sphere having the same volume $\frac{4}{3}\pi\bar{r}^3$, the quantity $\bar{r} = (ab^2)^{\frac{1}{3}}$ being an effective 'radius' of the prolate spheroid. If the velocity of the spheroid is $u(t) = u_0\cos\omega t$, (4.15) may be written as

$$F_z = D_0[-(k_1N_S + k_2\surd N_S)\sin\omega t + (1 + k_2\surd N_S)\cos\omega t], \tag{6.1}$$

where $D_0 = -8\pi\mu a u_0/\lambda_0\kappa$ is the steady-state drag for the spheroid at velocity u_0; N_S, sometimes called the Stokes number, is

$$(ab^2)^{\frac{2}{3}}\omega/\nu = \bar{r}^2\omega/\nu;$$

$$k_1 = \tfrac{1}{6}C_A\lambda_0\kappa[(\lambda_0^2-1)/\lambda_0^2]^{\frac{1}{3}} \quad \text{and} \quad k_2 = 2\surd 2[(\lambda_0^2-1)/\lambda_0^2]^{\frac{1}{6}}/3\lambda_0\kappa.$$

The characteristics of the force on the oscillating spheroid may be illustrated by non-dimensionalizing and rewriting (6.1) as

$$F_z/D_0 = K\cos(\omega t + \theta), \tag{6.2}$$

where K is the ratio of the amplitude of the drag on the oscillating prolate spheroid to the drag on the same spheroid moving at constant velocity u_0, and θ is the phase shift between the velocity of the spheroid and the force on the spheroid. Thus, from (6.1)

$$K = [(k_1N_S + k_2\surd N_S)^2 + (1 + k_2\surd N_S)^2]^{\frac{1}{2}}, \tag{6.3}$$

and

$$\theta = \tan^{-1}\left(\frac{k_1N_S + k_2\surd N_S}{1 + k_2\surd N_S}\right). \tag{6.4}$$

The amplitude ratio K and the phase shift θ are plotted against N_S for various values of a/b in figure 1.

For the case of an oblate spheroid, the dimensionless number N_S is defined as $(a^2b)^{\frac{2}{3}}\,\omega/\nu = \bar{r}^{*2}\omega/\nu$. The equations satisfied by the amplitude ratio K and the phase shift θ are the same as (6.3) and (6.4), respectively, with

$$k_1^* = \tfrac{1}{6}C_a^*\lambda_0^*\kappa^*[(\lambda_0^{*2}+1)/\lambda_0^*]^{\frac{1}{3}} \quad \text{and} \quad k_2^* = \tfrac{1}{3}2\sqrt{2}[(\lambda_0^{*2}+1)/\lambda_0^{*2}]^{-\frac{1}{3}}\lambda_0^*\kappa^*.$$

The amplitude ratio and the phase shift for the oblate spheroid are plotted against the Stokes number for various values of a/b in figure 2.

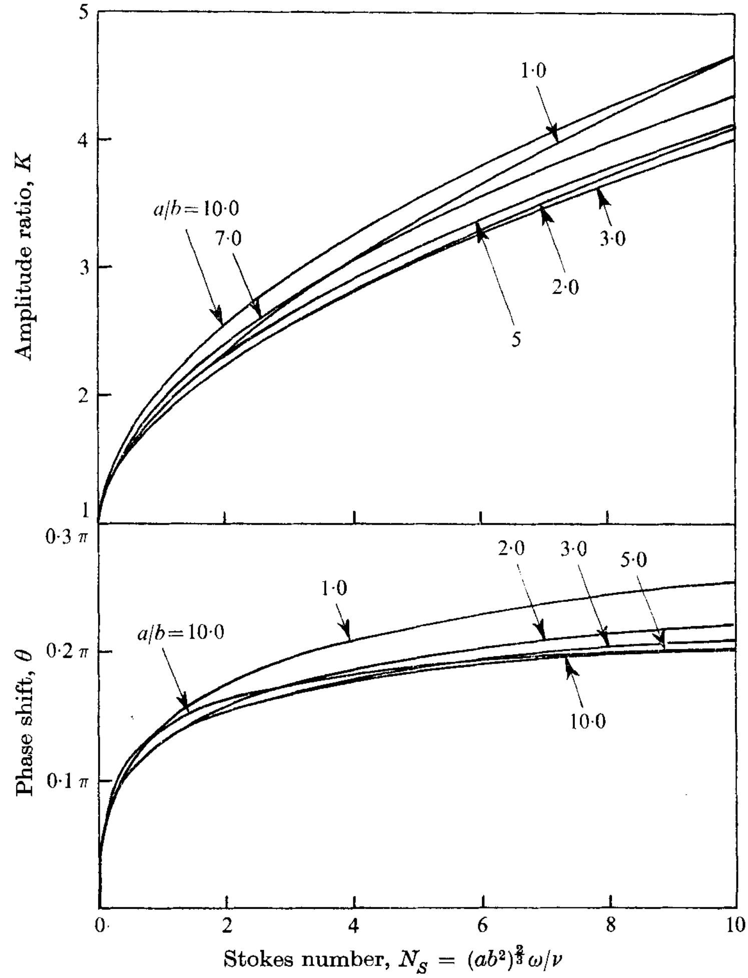

FIGURE 1. The ratio of the amplitude of the drag on an oscillating prolate spheroid to the drag on the same spheroid moving at constant velocity, K, and the phase shift between the velocity of the spheroid and the force on the spheroid, θ, both plotted against the Stokes number N_S.

7. The drag on a spheroid moving with an arbitrary velocity along its axis of symmetry

Landau & Lifshitz (1959, p. 96) calculated the resistance force, which is the identical to the result obtained by Basset (1888), for a sphere moving rectilinearly with arbitrary speed in an incompressible viscous fluid by integrating the drag on an oscillating sphere over all possible frequencies. This technique can be similarly used to calculate the drag on a spheroid moving with an arbitrary velocity along its axis of symmetry.

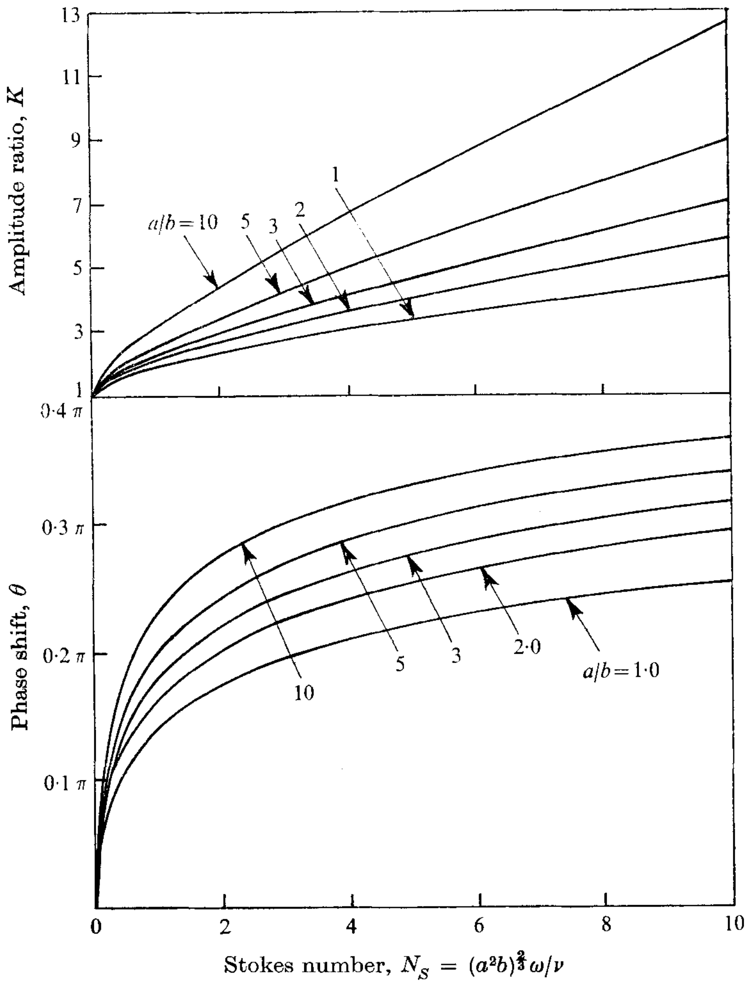

FIGURE 2. The ratio of the amplitude of the drag on an oscillating oblate spheroid to the drag on the same spheroid moving at constant velocity, K, and the phase shift between the velocity of the spheroid and the force on the spheroid, θ, both plotted against the Stokes number N_S.

In the previous sections the velocity of the spheroid is periodic, i.e.

$$u(t) = u_0 e^{-i\omega t}, \tag{7.1}$$

where u_0 is a constant and ω is the frequency. If the velocity of the spheroid is not periodic but arbitrary, the solution can be constructed from that obtained in the previous sections, provided the velocity can be represented as the Fourier integral

$$u(t) = \int_{-\infty}^{\infty} u_\omega e^{-i\omega t}\, d\omega, \quad u_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\tau)\, e^{i\omega\tau}\, d\tau, \tag{7.2}$$

where u_ω is the Fourier transform of $u(t)$. Since the equations involved in this problem are linear, the total drag on the accelerating spheroid can be written as the integral of the drag on the spheroid with velocities that are the separate Fourier components $u_\omega e^{-i\omega t}$. Thus the total drag corresponding to a velocity $u(t)$ given in (7.2) is

$$F_z(t) = \int_{-\infty}^{\infty} F_\omega(t, \omega)\, d\omega, \tag{7.3}$$

in which $F_\omega(t, \omega)$ is the drag corresponding to a velocity of $u_\omega e^{-i\omega t}$.

If the approximate drag given in (4.15) is used then $F_\omega(t, \omega)$ can be written as

$$F_\omega(t, \omega) = -\operatorname{Re}\left\{\tfrac{4}{3}\pi a b^2 \rho \left[\frac{(\lambda_0^2-1)\,Q_1(\lambda_0)}{1-(\lambda_0^2-1)\,Q_1(\lambda_0)}\right]\left(\frac{du}{dt}\right)_\omega e^{-i\omega t} + \frac{8\pi\mu a}{\lambda_0 \kappa} u_\omega e^{-i\omega t} + \frac{16\pi\rho b^2}{3(\lambda_0^2-1)\kappa^2}\,\frac{(1+i)(2\nu)^{\frac{1}{2}}}{\omega^{\frac{1}{2}}}\left(\frac{du}{dt}\right)_\omega e^{-i\omega t}\right\}, \tag{7.4}$$

where $(du/dt)_\omega = -i\omega u_\omega$. The substitution of (7.4) into (7.3) yields

$$F_z(t) = -\tfrac{4}{3}\pi a b^2 \rho \left[\frac{(\lambda_0^2-1)\,Q_1(\lambda_0)}{1-(\lambda_0^2-1)\,Q_1(\lambda_0)}\right]\frac{du(t)}{dt} - \frac{8\pi\mu a u(t)}{\lambda_0\kappa} - \operatorname{Re}\left\{\frac{16\pi\rho b^2 (2\nu)^{\frac{1}{2}}}{3(\lambda_0^2-1)\kappa^2}\int_{-\infty}^{\infty}\frac{(1+i)}{\omega^{2}}\left(\frac{du}{dt}\right)_\omega e^{-i\omega t}\, d\omega\right\}. \tag{7.5}$$

The last integral is found as (Landau & Lifshitz 1959, p. 96; Yih 1969, p. 376)

$$\operatorname{Re}\left\{\int_{-\infty}^{\infty}\frac{(1+i)}{\omega^{\frac{1}{2}}}\left(\frac{du}{dt}\right)_\omega e^{-i\omega t}\, d\omega\right\} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{du(\tau)/d\tau}{(t-\tau)^{\frac{1}{2}}}\, d\tau. \tag{7.6}$$

The insertion of (7.6) into (7.5) gives the approximate drag on a prolate spheroid moving along its axis of symmetry with arbitrary velocity through an incompressible viscous fluid:

$$F_z(t) = -\tfrac{4}{3}\pi a b^2 \rho \left[\frac{(\lambda_0^2-1)\,Q_1(\lambda_0)}{1-(\lambda_0^2-1)\,Q_1(\lambda_0)}\right]\frac{du}{dt} - \frac{8\pi\mu a u}{\lambda_0\kappa} - \frac{32\pi b^2 \rho}{3(\lambda_0^2-1)\kappa^2}\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{du(\tau)/d\tau}{(t-\tau)^{\frac{1}{2}}}\, d\tau. \tag{7.7}$$

In (7.7) the fluid resistance consists of three parts. The first is the added mass effect, the second is the steady-state drag and the last is the effect due to the history of the motion. The constant $C_H = 32/3(\lambda_0^2-1)\,\kappa^2$ may be defined as the history coefficient for the prolate spheroid accelerating along its axis of symmetry.

For an oblate spheroid moving with arbitrary velocity along its axis of symmetry in a viscous fluid the approximate drag may be found either by integrating (5.1) with respect to the frequency or by substituting $i\lambda_0$ for λ_0 into (7.7). The result is

$$F_z(t) = -\tfrac{4}{3}\pi a^2 b \rho \left[\frac{(\lambda_0^{*2}+1)\,q_1(\lambda_0^*)}{1-(\lambda_0^{*2}+1)\,q_1(\lambda_0^*)}\right]\frac{du}{dt} - \frac{8\pi\mu b u}{\lambda_0^*\kappa^*} - \frac{32\pi a^2 \rho}{3(\lambda_0^{*2}+1)\kappa^{*2}}\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{du(\tau)/d\tau}{(t-\tau)^{\frac{1}{2}}}\, d\tau, \tag{7.8}$$

and the quantity $C_H^* = 32/3(\lambda_0^{*2}+1)\,\kappa^{*2}$ may be defined as the history coefficient for the oblate spheroid accelerating along its axis of symmetry.

The expression for the drag on a circular disk accelerating along the direction normal to its surface is obtained by letting $\lambda_0^* \to 0$ in (7.8). The result is

$$F_z(t) = -\tfrac{8}{3}a^3\rho\frac{du}{dt} - 16\mu a u - \frac{128a^2\rho}{3\pi}\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{du(\tau)/d\tau}{(t-\tau)^{\frac{1}{2}}}\,d\tau. \qquad (7.9)$$

The history coefficient for the circular disk is $C_H = 128/3\pi^2$. When the spheroid becomes a sphere, i.e. $\lambda_0 \to \infty$, (7.7) and (7.8) both approach the result obtained by Basset (1888) and by Landau & Lifshitz (1959).

	Prolate spheroid			Oblate spheroid		
a/b	C_A	C_V	C_H	C_A^*	C_V^*	C_H^*
1·0	0·5000	1·0000	6·0000	0·5000	1·0000	6·0000
1·5	0·3037	1·1017	7·2820	0·8047	0·9352	5·2479
2·0	0·2100	1·2039	8·6968	1·1151	0·9053	4·9175
3·0	0·1220	1·4045	11·8351	1·7426	0·8787	4·6325
4·0	0·0816	1·5979	15·3205	2·3743	0·8673	4·5137
5·0	0·0591	1·7848	19·1133	3·0078	0·8615	4·4526
6·0	0·0452	1·9659	23·1895	3·6422	0·8580	4·4169
7·0	0·0358	2·1421	27·5324	4·2772	0·8558	4·3942
8·0	0·0293	2·3141	32·1291	4·9126	0·8543	4·3788
9·0	0·0244	2·4822	36·9692	5·5483	0·8532	4·3680
10·0	0·0207	2·6471	42·0440	6·1841	0·8525	4·3600
∞	0	∞	∞	∞	0·8488	4·3231

TABLE 1. Added mass, viscous and history coefficients for some prolate and oblate spheroids

Finally, it is convenient to write the approximate drag on a spheroid moving in an arbitrary manner along its axis of symmetry in the following form:

$$F_z(t) = -mC_A\frac{du}{dt} - C_V\,6\pi\mu R u - C_H A\rho\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{t}\frac{du(\tau)/d\tau}{(t-\tau)^{\frac{1}{2}}}\,d\tau, \qquad (7.10)$$

where m is the mass of displaced fluid, C_A is the added mass coefficient, C_V is the viscous shape coefficient for Stokes steady-state drag, R is dimension b for the prolate spheroid and dimension a for the oblate spheroid, C_H is the history coefficient and A is the cross-sectional area normal to the motion. For some representative values of a/b, C_A, C_V and C_H for prolate and oblate spheroids are given in table 1.

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